arXiv:0902.3298v1 [math.CA] 19 Feb 2009

SHARPENING AND GENERALIZATIONS OF SHAFER'S INEQUALITY FOR THE ARC TANGENT FUNCTION

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Abstract. In this paper, we sharpen and generalize Shafer's inequality for the arc tangent function. From this, some known results are refined.

1. Introduction and main results

In [8], the following elementary problem was posed: Show that for $x>0$

$$\arctan x > \frac{3x}{1+2\sqrt{1+x^2}}. \tag{1}$$

In [9], the following three proofs for the inequality (1) were provided.

Solution by Grinstein: Direct computation gives

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = \frac{\bigl(\sqrt{1+x^2}-1\bigr)^2}{\bigl(1+x^2\bigr)\bigl(1+2\sqrt{1+x^2}\bigr)^2},$$

where

$$F(x) = \arctan x - \frac{3x}{1+2\sqrt{1+x^2}}.$$

Now $\frac{\mathrm{d}F(x)}{\mathrm{d}x}$ is positive for all $x\ne 0$, whence $F(x)$ is an increasing function. Since $F(0)=0$, it follows that $F(x)>0$ for $x>0$.

Solution by Marsh: It follows from $(\cos\phi-1)^2\ge 0$ that

$$1 \ge \frac{3+6\cos\phi}{(\cos\phi+2)^2}.$$

The desired result is obtained directly upon integration of the latter inequality with respect to ϕ from 0 to $\arctan x$ for $x>0$.

Solution by Konhauser: The substitution $x=\tan y$ transforms the given inequality into $y>\frac{3\sin y}{2+\cos y}$, which is a special case of an inequality discussed on [4, pp. 105–106].

It may be worthwhile to note that the inequality (1) is not collected in the authorized monograph [2] and [3].

In [2, pp. 288–289], the following inequalities for the arc tangent function are collected:

$$\arctan x < \frac{2x}{1+\sqrt{1+x^2}}, \tag{2}$$

2000 *Mathematics Subject Classification.* Primary 33B10; Secondary 26D05.

Key words and phrases. sharpening, generalization, Shafer's inequality, arc tangent function, monotonicity.

The first author was partially supported by the China Scholarship Council.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

$$\frac{x}{1+x^2} < \arctan x < x, \tag{3}$$

$$x - \frac{x^3}{3} < \arctan x < x, \tag{4}$$

$$\frac{1}{2x}\ln\bigl(1+x^2\bigr) < \arctan x < (1+x)\ln(1+x), \tag{5}$$

where $x > 0$.

The aim of this paper is to sharpen and generalize inequalities (1) and (2).

Our results may be stated as the following theorems.

Theorem 1. *For $x > 0$, let*

$$f_a(x) = \frac{\bigl(a+\sqrt{1+x^2}\,\bigr)\arctan x}{x}, \tag{6}$$

where a is a real number.

1. *When $a \le -1$ or $0 \le a \le \frac{1}{2}$, the function $f_a(x)$ is strictly increasing on $(0,\infty)$;*
2. *When $a \ge \frac{2}{\pi}$, the function $f_a(x)$ is strictly decreasing on $(0,\infty)$;*
3. *When $\frac{1}{2} < a < \frac{2}{\pi}$, the function $f_a(x)$ has a unique minimum on $(0,\infty)$.*

As direct consequences of Theorem 1, the following inequalities may be derived.

Theorem 2. *For $0 \le a \le \frac{1}{2}$,*

$$\frac{(1+a)x}{a+\sqrt{1+x^2}} < \arctan x < \frac{(\pi/2)x}{a+\sqrt{1+x^2}}, \quad x > 0. \tag{7}$$

For $\frac{1}{2} < a < \frac{2}{\pi}$,

$$\frac{4a\bigl(1-a^2\bigr)x}{a+\sqrt{1+x^2}} < \arctan x < \frac{\max\{(\pi/2), 1+a\}x}{a+\sqrt{1+x^2}}, \quad x > 0. \tag{8}$$

For $a \ge \frac{2}{\pi}$, the inequality (7) is reversed.

Moreover, the constants $1+a$ and $\frac{\pi}{2}$ in inequalities(7) and (8) are the best possible.

2. Remarks

Before proving our theorems, we are about to give several remarks on them.

Remark 1. The inequality (1) is the special case $a = \frac{1}{2}$ of the left-hand side inequality in (7).

Remark 2. The inequality (2) is the special case $a = 1$ of the reversed version of the left hand-side inequality in (7).

Remark 3. The inequality (2) is better than (5). If taking $a = \frac{2}{\pi}$ in (7), then

$$\frac{\pi^2 x}{2+2\pi\sqrt{1+x^2}} < \arctan x < \frac{(\pi+2)x}{2+\pi\sqrt{1+x^2}}, \quad x > 0. \tag{9}$$

This double inequality refines corresponding ones in (1), (2), (3), (4) and (5).

Remark 4. The substitution $x = \tan y$ may transform inequalities in (7) and (8) into some trigonometric inequalities.

Remark 5. The approach below used in the proofs of Theorem 1 and Theorem 2 has been employed in [1, 5, 6, 7].

3. Proofs of theorems

Now we are in a position to prove our theorems.

Proof of Theorem 1. Direct calculation gives

$$\begin{aligned}
f_a'(x) &= \frac{\left(1+x^2\right)\left(1+a\sqrt{1+x^2}\,\right)}{x^2\left(1+x^2\right)^{3/2}}\left[\frac{x+x^3+ax\sqrt{1+x^2}}{\left(1+x^2\right)\left(1+a\sqrt{1+x^2}\,\right)}-\arctan x\right]\\
&\triangleq \frac{\left(1+x^2\right)\left(1+a\sqrt{1+x^2}\,\right)}{x^2\left(1+x^2\right)^{3/2}}g_a(x),\\
g_a'(x) &= -\frac{x^2\left(2a^2\sqrt{x^2+1}\,+a-\sqrt{x^2+1}\,\right)}{\left(x^2+1\right)^{3/2}\left(a\sqrt{x^2+1}\,+1\right)^2}\\
&\triangleq -\frac{x^2h_a(x)}{\left(x^2+1\right)^{3/2}\left(a\sqrt{x^2+1}\,+1\right)^2},
\end{aligned}$$

and the function $h_a(x)$ has two zeros

$$a_1(x)=-\frac{1+\sqrt{9+8x^2}}{4\sqrt{1+x^2}}\quad\text{and}\quad a_2(x)=\frac{-1+\sqrt{9+8x^2}}{4\sqrt{1+x^2}}.$$

Further differentiation yields

$$a_1'(x)=\frac{x\left(1+\sqrt{9+8x^2}\,\right)}{4(1+x^2)^{3/2}\sqrt{9+8x^2}}>0$$

and

$$a_2'(x)=\frac{x\left(\sqrt{9+8x^2}\,-1\right)}{4(1+x^2)^{3/2}\sqrt{9+8x^2}}>0.$$

This means that the functions $a_1(x)$ and $a_2(x)$ are increasing on $(0,\infty)$. From

$$\lim_{x\to0^+}a_1(x)=-1,\qquad \lim_{x\to\infty}a_1(x)=-\frac{\sqrt2}{2},$$
$$\lim_{x\to0^+}a_2(x)=\frac12,\qquad \lim_{x\to\infty}a_2(x)=\frac{\sqrt2}{2},$$

it follows that

(1) when $a\le-1$ or $a\ge\frac{\sqrt2}{2}$, the derivative $g_a'(x)$ is negative and the function $g_a(x)$ is strictly decreasing on $(0,\infty)$. From

$$\lim_{x\to0^+}g_a(x)=0\quad\text{and}\quad\lim_{x\to\infty}g_a(x)=\frac1a-\frac\pi2, \tag{10}$$

it is deduced that $g_a(x)<0$ on $(0,\infty)$. Accordingly,
   - (a) when $a\le-1$, the derivative $f_a'(x)>0$ and the function $f_a(x)$ is strictly increasing on $(0,\infty)$;
   - (b) when $a\ge\frac{\sqrt2}{2}$, the derivative $f_a'(x)$ is negative and the function $f_a(x)$ is strictly decreasing on $(0,\infty)$.

(2) when $\frac12\ge a\ge0$, the derivative $g_a'(x)$ is positive and the function $g_a(x)$ is increasing on $(0,\infty)$. By (10), it follows that the function $g_a(x)$ is positive on $(0,\infty)$. Thus, the derivative $f_a'(x)$ is positive and the function $f_a(x)$ is strictly increasing on $(0,\infty)$.

(3) when $\frac{1}{2} < a < \frac{\sqrt{2}}{2}$, the derivative $g_a'(x)$ has a unique zero which is a minimum of $g_a(x)$ on $(0,\infty)$. Hence, by the second limit in (10), it may be deduced that
   (a) when $\frac{2}{\pi} \le a < \frac{\sqrt{2}}{2}$, the function $g_a(x)$ is negative on $(0,\infty)$, so the derivative $f_a'(x)$ is also negative and the function $f_a(x)$ is strictly decreasing on $(0,\infty)$;
   (b) when $\frac{1}{2} < a < \frac{2}{\pi}$, the function $g_a(x)$ has a unique zero which is also a unique zero of the derivative $f_a'(x)$, and so the function $f_a(x)$ has a unique minimum of the function $f_a(x)$ on $(0,\infty)$.

The proof of Theorem 1 is complete. □

Proof of Theorem 2. Direct calculation yields

$$\lim_{x\to0^+} f_a(x) = 1+a \quad \text{and} \quad \lim_{x\to\infty} f_a(x) = \frac{\pi}{2}.$$

By the increasing monotonicity in Theorem 1, it follows that $1+a < f_a(x) < \frac{\pi}{2}$ for $0 \le a \le \frac{1}{2}$, which can be rewritten as (7). Similarly, the reversed version of the inequality (7) and the right-hand side inequality in (8) can be procured.

When $\frac{1}{2} < a < \frac{2}{\pi}$, the unique minimum point $x_0 \in (0,\infty)$ of the function $f_a(x)$ satisfies

$$\arctan x_0 = \frac{x_0 + x_0^3 + ax_0\sqrt{1+x_0^2}}{\bigl(1+x_0^2\bigr)\bigl(1+a\sqrt{1+x_0^2}\,\bigr)},$$

and so the minimum of $f_a(x)$ on $(0,\infty)$ is

$$\begin{aligned} f_a(x_0) &= \frac{x_0 + x_0^3 + ax_0\sqrt{1+x_0^2}}{\bigl(1+x_0^2\bigr)\bigl(1+a\sqrt{1+x_0^2}\,\bigr)} \cdot \frac{a+\sqrt{1+x_0^2}}{x_0} \\ &= \frac{\bigl(a+\sqrt{1+x_0^2}\,\bigr)\bigl(1+x_0^2+a\sqrt{1+x_0^2}\,\bigr)}{\bigl(1+x_0^2\bigr)\bigl(1+a\sqrt{1+x_0^2}\,\bigr)} \\ &= \frac{(a+u)^2}{u\bigl(1+au\bigr)}, \quad u=\sqrt{1+x_0^2} \in (1,\infty) \\ &> 4a\bigl(1-a^2\bigr), \end{aligned}$$

as a result, the left-hand side inequality in (8) follows. The proof of Theorem 2 is complete. □

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